Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(01)00049-1

J. Appl. Maths Mechs, Vol. 65, No. 3, pp. 435–440, 2001

0021-8928/01/\$—see front matter

HOLONOMIC CONSTRAINTS AND SOME EXACT SOLUTIONS OF THE EQUATIONS OF STEADY ONE-DIMENSIONAL GAS FLOW†

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(*Received* 28 *February* 2000)

A first-order quasi-linear homogeneous system of equations with two dependent and two independent variables is considered. Various one-dimensional steady gas flows can be modelled by systems of this type. The symmetries of the system considered are used to construct holonomic constraints between the dependent (and independent) variables. The holonomic constraints between the variables are used to reduce the dimension of the system and to construct its exact solutions. © 2001 Elsevier Science Ltd. All rights reserved.

It is well known that a knowledge of the symmetries (groups of continuous transformations) of systems of partial differential equations enables one to simplify the problem of integrating them. Various possibilities of using symmetries to integrate systems of partial differential equations have been considered, for example, by Ovsyannikov [1] and Olver [2].

It is also well known that a knowledge of the symmetries of a system of ordinary differential equations enables its first integrals to be found. The first integral can be regarded as a certain holonomic equation of the constraint between the dependent variables of the system. A knowledge of this constraint for a system of equations enables its dimension to be reduced by one. The possibilities of a similar approach to the investigation of symmetries were considered earlier (see, for example, [3]).

The notion of a first integral has been used for various purposes for systems of partial differential equations as well. In particular, Bernoulli's equation of the steady motion of an ideal gas is obtained as a first integral (along streamlines) of the equations of gas motion [4]. The notion of a first integral for a system of partial differential equations has been used to solve some variational problems [5].

Instead of the notion of a "first integral" we will use the term "holonomic constraint" between the variables of the system. In the general case, this constraint contains the independent variables of the system in addition to the dependent variables. The use of such a constraint for systems of partial differential equations, unlike systems of ordinary differential equations, leads to limitations in formulating the boundary conditions for the system. Hence, there are also limitations on the form of the solutions (the integral manifolds of the system).

In this paper we use the notion of a holonomic constraint to reduce the dimension and to construct exact solutions of a first-order quasi-linear homogeous system of partial differential equations with two independent and two dependent variables. Some gas and liquid flow processes are described by systems of this form, for example, the one-dimensional steady isoentropic gas flow and the liquid flow in a pipeline with elastic walls.

Note also that a holonomic constraint can be regarded as a special form of constraint of more general form, which also includes the derivatives of the dependent variables. These constraints are constructed using the method of differential constraints, a description of which can be found in [6]. The addition of a differential constraint or a holonomic constraint between dependent variables makes the initial system overdefined. Hence, the main "techniques" of the method of differential constraints is an investigation of the overdefined system for compatibility using Cartan's method of external forms or the Janet Spencer–Kuranishi method.

In this paper, to construct holonomic constraints we will use certain "a priori information", starting from a preliminary analysis of the properties of the initial system. That is, to construct the holonomic constraint of the system of equations considered we will use the symmetries of the system.

†*Prikl. Mat. Mekh.* Vol. 65, No. 3, pp. 449–455, 2001.

1. THE INITIAL SYSTEM OF EQUATIONS AND THE DETERMINATION OF THE HOLONOMIC CONSTRAINT

Consider the system of equations

$$\frac{\partial u^1}{\partial x^1}+\alpha_{11}u^1\frac{\partial u^1}{\partial x^2}+\alpha_{12}u^2\frac{\partial u^2}{\partial x^2}=0,\quad \frac{\partial u^2}{\partial x^1}+\alpha_{21}u^2\frac{\partial u^1}{\partial x^2}+\alpha_{22}u^1\frac{\partial u^2}{\partial x^2}=0 \tag{1.1}$$

Here x^1 and x^2 are the independent variables, u^1 and u^2 are the dependent variables, and the coefficients α_{11}, α_{12}, α_{21} and α_{22} are constants.

Note that, for systems of equations which describe the one-dimensional steady fluid flow, x^1 defines the time while x^2 defines the spatial coordinate. The variable u^1 usually defines the longitudinal flow rate, while the physical meaning of the variable u^2 depends on the problem considered. For the one-dimensional steady isoentropic gas flow, u^2 defines the velocity of sound in the flow [7]. For the liquid flow in an elastic pipeline, u^2 defines cross-section area of the pipeline [8]. The matrix of the coefficients of the system is also governed by the specific form of the problem.

We will denote by $E \subset R^4$ the subspace formed by the dependent and independent variables of the system. In the geometrical approach, system (1.1) can be regarded as a certain surface Σ in space J^1, consisting of the variables of the system and the derivatives of the dependent variables (1-jet space)[9]. The space of the integral manifolds of system Σ will be denoted by U_Σ.

Consider a certain function $\phi(u, x)$ and a certain subspace $U_s \subset U_\Sigma$.

Definition 1. We will call the expression

$$\phi(u,x)\big|_{U_s}=c \quad (c=\text{const}) \tag{1.2}$$

a holonomic constraint between the dependent (and also, in general, the independent) variables of system of equations (1.1) in the subspace of integral manifolds U_s. (For simplicity we will also use the abbreviated designation HC.)

According to this definition, it is not required that the function $\phi(u, x)$ should define HC (1.2) over the whole space U_Σ (as has occurred for systems of ordinary differential equations). This means that when seeking an HC it is necessary to determine, in addition to the function $\phi(u, x)$ itself, a subspace U_s also, for which relation (1.2) is satisfied.

The use of an HC enables one to reduce the dimension of the system, but impose limitations on the space of the integral manifolds of system (1.1). Hence, below we give a definition of two "varieties" of HC.

Definition 2. We will call relation (1.2) the complete HC of system (1.1) if the subspace U_s is defined by the solutions of a certain partial differential equation with two independent variables.

Definition 3. We will call relation (1.2) a particular HC of system (1.1) if the subspace U_s is defined by the solutions of an ordinary differential equation (or a system of ordinary differential equations) and is defined apart from a certain set of constants.

Holonomic constraints of this kind are called particular HCs since they can be determined immediately if we know some analytical solution of system (1.1). Such HCs will be of no interest here. However, HCs of this kind will be considered if they appear when solving the problem of finding the function $\phi(u, x)$ because of limitations on the space U_Σ, for which the given specific function $\phi(u, x)$ exists.

2. THE CONDITIONS FOR A HOLONOMIC CONSTRAINT TO EXIST

As noted above, when solving the problem of finding the first integral for a system of ordinary differential equations, a group-theory approach is used in a number of papers.

We will use the terminology of [9] below: the Lie vector field $\overline{X}$ is called an infinitesimal symmetry of system (1.1) if it touches the surface Σ. Note that this condition is equivalent [1] to the condition that system (1.1) should be invariant under the action of the continued operator $\overline{X}$ of a continuous group of transformations.

By the terminology of [9], when seeking the first integral in [3] a function is used which is obtained from the condition that a certain surface should be touched by the vector field $\overline{X}$. In this paper we will also define the HCs for a system of partial differential equations using the group-theory approach.

Suppose the vector field $\overline{X}$ is an infinitesimal symmetry of system (1.1). We will denote by $\overline{X}$ the projection of the vector field $\overline{X}$ onto the space E. We define the function $\phi(u, x)$ (or a family of functions) from the condition

$$\overline{X}(\phi(u,x)) = X(\phi(u,x)) = 0 \tag{2.1}$$

We will further use the function $\phi(u,x)$, which satisfies condition (2.1), to find the HC. To do this we determine the family of "invariant" surfaces S_e in E-space from the condition that the vector field X touches these surfaces. This family of surfaces can be defined by the expression

$$\phi(u,x) = c \quad (c = \text{const}) \tag{2.2}$$

If relation (2.2) holds for a certain function, which satisfies conditions (2.1) and a certain subspace U_s of integral manifolds of system (1.1), then, by Definitions 1–3, the function $\phi(u,x)$ defines the HC.

We can also give the following interpretation of the use of the vector field $\overline{X}$. It follows from the above that the vector field $\overline{X}$ is touched by the surfaces S_e and Σ. Differentiating expression (2.2) with respect to the independent variable, we obtain in J^1 a certain surface S_e^1, which is the "transform" of the surface S_e. For an HC (some "degree of completeness") to exist, it is necessary that the surfaces S_e^1 and Σ should intersect.

Hence, we need to determine the specific form of the function $\phi(u,x)$ and the subspace U_s for this function.

Suppose expression (2.2) is solvable for one of the variables u^1, i.e. we can write

$$u^l = \phi_{\bar{l}}^{-1}(u^{\bar{l}}, x^1, x^2) \tag{2.3}$$

where $\phi_{\bar{l}}^{-1}(u^{\bar{l}}, x^1, x^2)$ is the "inverse" function, obtained when solving (2.2) for u^l. Here $\bar{l} = 1$ if $l = 2$ and $\bar{l} = 2$ if $l = 1$.

We substitute the expression for the variable u^l from (2.3) into (1.1). In the general case we obtain an overdefined system of partial differential equations Σ_ϕ, containing two equations in the variable $u^{\bar{l}}$. We will assume that system Σ_ϕ is compatible if at least one function $u^{\bar{l}}(x^1, x^2)$ (not identically constant) exists which is the solution of system Σ_ϕ.

Two cases are possible for the system of equations of the form considered. Suppose one of the equations of system Σ_ϕ depends linearly on the other equation. Then, from a comparison of these equations no limitations (constraints) follow for the variable occurring in $\phi_{\bar{l}}^{-1}(u,x)$, and we can regard Σ_ϕ has an equation with a single dependent variable $u^{\bar{l}}$. The boundary conditions for this partial differential equation can be specified with a "functional" arbitrariness and, consequently, the function $\phi(u,x)$ defines the complete HC.

If, on substituting $\phi_{\bar{l}}^{-1}(u,x)$ into (1.1), the equations of system Σ_ϕ are linearly independent, then system Σ_ϕ is overdefined and when it is compatible we obtain a particular HC.

In the light of the above we can formulate the following assertion.

Assertion. If on substituting $\phi_{\bar{l}}^{-1}(u,x)$ into (1.1) the equations of system Σ_ϕ are linearly dependent, HC (2.2) exists and is complete. If the equations of system Σ_ϕ are not linearly dependent, then when Σ_ϕ is compatible we have a particular HC.

Note that for the system considered, which has only two independent variables, we can have HCs corresponding to only two "limiting" cases (i.e. Definition 2 or Definition 3). For systems of equations of a different class other HCs (of an "intermediate" type) are also possible.

We will now construct the HC of system (1.1) using the above definitions and procedure.

3. HOLONOMIC CONSTRAINTS AND THE SOLUTIONS OF THE SYSTEM

The governing equations for the infinitesimal symmetries of system (1.1) were obtained in [10]. It follows from the solution of these equations that the infinitesimal symmetries of system (1.1) are defined by vector fields X of the form

$$X_1 = \frac{\partial}{\partial x^1}, \quad X_2 = \frac{\partial}{\partial x^2}, \quad X_3 = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2}$$

$$X_4 = u^1 \frac{\partial}{\partial u^1} + u^2 \frac{\partial}{\partial u^2} + x^2 \frac{\partial}{\partial x^2}, \quad X_5 = \frac{\partial}{\partial u^1} + x^1 \frac{\partial}{\partial x^2}$$

We will construct the HCs for some vector fields.

Vector field X_1. The vector field X_1 touches the family of surfaces

$$\phi(u^1, u^2, x^2) = \text{const}$$

where ϕ is an arbitrary continuous function. Since the function ϕ is arbitrary, we will seek an HC of the form

$$u^2 = \psi(x^2, u^1) \tag{3.1}$$

where $\psi(x^2, u^1)$ is an arbitrary function ($\psi(x^2, u^1) = \phi_2^{-1}(u^1, u^2, x^2)$).
Substituting (3.1) into (1.1) we obtain the system Σ_2

$$\frac{\partial u^1}{\partial x^1} + \alpha_{11} u^1 \frac{\partial u^1}{\partial x^2} + \alpha_{12} \psi \frac{\partial \psi}{\partial x^2} + \alpha_{12} \psi \frac{\partial \psi}{\partial u^1} \frac{\partial u^1}{\partial x^2} = 0$$

$$\frac{\partial \psi}{\partial u^1} \frac{\partial u^1}{\partial x^1} + \alpha_{22} u^1 \frac{\partial \psi}{\partial x^2} + \alpha_{22} u^1 \frac{\partial \psi}{\partial u^1} \frac{\partial u^1}{\partial x^2} + \alpha_{21} \psi \frac{\partial u^1}{\partial x^2} = 0$$

We will seek the function $\psi(x^2, u^1)$ such that the HC is complete (by Definition 2). To do this we assume that the function ψ depends solely on u^1. After substitution we can conclude that system Σ_2 consists of linearly dependent equations with $\psi = k_1 u^1 + k_2$, where

$$k_1 = [(\alpha_{22} + \alpha_{21} - \alpha_{11}) / \alpha_{12}]^{1/2}$$

The constant $k_2 = 0$ when $\alpha_{11} \neq \alpha_{22}$ and k_2 is an arbitrary constant when $\alpha_{11} = \alpha_{22}$. For $\psi = k_1 u^1 + k_2$ we obtain the system Σ_ϕ from system Σ_2, which in this case consists of a single equation

$$\frac{\partial u^1}{\partial x^1} + \left(\alpha_{22} u^1 + \alpha_{21} u^1 + \alpha_{21} \frac{k_2}{k_1} \right) \frac{\partial u^1}{\partial x^2} = 0 \tag{3.2}$$

Hence, the complete HC for system (1.1), constructed using vector field X_1, has the form of a surface

$$u^2 - k_1 u^1 = k_2 \tag{3.3}$$

Here the subspace U_s of the integral manifolds of system (1.1) is determined by the solutions of Eq. (3.2) and formula (3.3).

Note that the HC of the form $u^2 = \psi(u^1)$ defines a travelling wave of the first kind (a simple wave). Simple waves for the one-dimensional equations of gas dynamics were discovered by Riemann, and for systems of multidimensional quasi-linear partial differential equations by Yanenko [11].

Particular solutions of Eq. (3.2) can be obtained using the expression

$$x^2 = (\alpha_{22} u^1 + \alpha_{21} u^1 + \alpha_{21} k_2 / k_1) x^1 + f(u^1)$$

where $f(u^1)$ is an arbitrary continuous function. By specifying the form of this function we obtain different solutions of the initial system.

A similar HC can be obtained using the vector field X_3.

The vector field X_2. The vector field X_2, like X_1, enables us to obtain HC (3.3). However, we will construct the particular HC using this vector field for the case when $\alpha_{11} = \alpha_{22} = 1$. Note that the one-dimensional steady isoentropic flow of a gas can be described by a system of this form [7].

The vector field X_2 touches the family of surfaces

$$\phi(u^1, u^2, x^1) = \text{const}$$

where ϕ is an arbitrary continuous function. Since the function ϕ is arbitrary, we can put

$$u^2 = \psi(x^1, u^1) \tag{3.4}$$

where $\psi(x^1, u^1)$ is an arbitrary function ($\psi(x^1, u^1) = \phi_2^{-1}(u^1, u^2, x^1)$).
Substituting (3.4) into (1.1) we obtain the system Σ_2

$$\frac{\partial u^1}{\partial x^1} + \left(u^1 + \alpha_{12}\psi\frac{\partial\psi}{\partial u^1}\right)\frac{\partial u^1}{\partial x^2} = 0, \quad \frac{\partial\psi}{\partial x^1} + \frac{\partial\psi}{\partial u^1}\frac{\partial u^1}{\partial x^1} + \left(u^1\frac{\partial\psi}{\partial u^1} + \alpha_{21}\psi\right)\frac{\partial u^1}{\partial x^2} = 0$$

From the condition for this system to be compatible we have

$$\frac{\partial\psi}{\partial x^1} = \psi\frac{\partial u^1}{\partial x^2}\left(\alpha_{12}\left(\frac{\partial\psi}{\partial u^1}\right)^2 - \alpha_{21}\right) \tag{3.5}$$

We will seek the function $\psi(x^1, u^1)$ such that the HC corresponds to Definition 3. To do this we can put

$$\psi = (u^1 + \gamma)^n f(x^1) \quad (\gamma = \text{const}, n = \text{const}) \tag{3.6}$$

where $f(x^1)$ is an arbitrary function. The HC can be obtained most simply for $n = 1$. In this case, from (3.5), taking (3.6) into account, we obtain the expression

$$u^1 = \frac{x^2}{\alpha_{12}f^3 - \alpha_{21}f}\frac{df}{dx^1} + S \tag{3.7}$$

where $S(x^1)$ is an arbitrary continuous function. Substituting (3.6) and (3.7) into system Σ_2 we obtain a system of ordinary differential equations Σ_ϕ for the functions $f(x^1)$ and $S(x^1)$

$$\frac{d^2 f}{(dx^1)^2}(\alpha_{12}f^3 - \alpha_{21}f) + \left(\frac{df}{dx^1}\right)^2 (1 + \alpha_{21} - 2\alpha_{12}f^2) = 0$$

$$\frac{dS}{dx^1}(\alpha_{12}f^3 - \alpha_{21}f) + \frac{df}{dx^1}(S + \alpha_{12}Sf^2 + \alpha_{12}\gamma f^2) = 0$$

(symbolic calculations were carried out on a computer using the MAPLE V program). The solution of this system (together with (3.4) and (3.6)–(3.7)) determines U_s.
Hence, the HC for system (1.1), constructed using the vector field X_2, has the form of a surface

$$u^2 - (u^1 + \gamma)f = 0 \tag{3.8}$$

In this case the subspace U_s of integral manifolds of system (1.1), for which this HC exists, is defined by the solutions of system Σ_ϕ and formula (3.7).
Note that the vector field X_4 also enables us to determine the HC of the form (3.8) with $\gamma = 0$, since

$$X_4(u^2 - u^1 f)\Big|_{u^2 = u^1 f} = 0$$

The vector field X_5. The vector field X_5 touches the family of surfaces

$$\phi(u^1 - x^2/x^1, u^2, x^1) = \text{const}$$

where ϕ is an arbitrary continuous function. Since the function ϕ is arbitrary, we can put

$$u^1 = x^2/x^1 + \psi(x^1, u^2) \tag{3.9}$$

where $\psi(x^1, u^2)$ is an arbitrary function.

The condition for the system Σ_1 to be compatible (when $\alpha_{11} = \alpha_{22} = 1$) has the form

$$\frac{\partial\psi}{\partial x^1} + \frac{\psi}{x^1} + \alpha_{12}u^2\frac{\partial u^2}{\partial x^2} = \alpha_{21}\frac{u^2}{x^1}\frac{\partial\psi}{\partial u^2} + \alpha_{21}u^2\left(\frac{\partial\psi}{\partial u^2}\right)^2\frac{\partial u^2}{\partial x^2} \tag{3.10}$$

This condition can be satisfied assuming

$$\psi = (u^2)^n f(x^1) \quad (n = \text{const}) \tag{3.11}$$

where $f(x^1)$ is an arbitrary function. The HC can be found most simply when $n = 1$. In this case, from (3.10), taking expression (3.11) into account, we have

$$u^2 = \frac{(\alpha_{21} - 1)fx^2}{x^1(\alpha_{12} - \alpha_{21}f^2)} - \frac{df}{dx^1}\frac{x^2}{(\alpha_{12} - \alpha_{21}f^2)} + S \tag{3.12}$$

where $S(x^1)$ is an arbitrary continuous function. For the functions $f(x^1)$ and $S(x^1)$ we have the system of ordinary differential equations Σ_φ

$$\frac{d^2f}{(dx^1)^2}(\alpha_{21}f^2 - \alpha_{12})x^1 + \frac{df}{dx^1}\left(x^1f\frac{df}{dx^1} + 2f^2 - \alpha_{21}x^1f\frac{df}{dx^1} - 2\alpha_{12}\right) +$$

$$+ f(1 - \alpha_{21})(f^2 - \alpha_{21}\alpha_{12}) = 0$$

$$\frac{dS}{dx^1}(\alpha_{21}f^2 - \alpha_{12})x^1 + S\left(\alpha_{21}fx^1\frac{df}{dx^1} + fx^1\frac{df}{dx^1} + f^2 - \alpha_{21}\alpha_{12}\right) = 0$$

The solution of these equations determines the subspace of integral manifolds U_s of system (1.1) for the HC

$$u^1 - x^2/x^1 - u^2f(x^1) = 0$$

where u^2 is defined by (3.12).

Hence, the symmetries of the system of partial differential equations can be used to construct holonomic constraints between the dependent (and independent) variables of the system. The use of these constraints enables the dimension of the system to be reduced and enables its particular exact solutions to be constructed.

I wish to thank K. G. Garayev and V. G. Pavlov for useful comments.

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Translated by R.C.G.